ON THE STABILITY OF SPINNING OF A TOP WITH A LIQUID FILLED CAVITY

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PMM Vol.29, № 1, 1965, pp.35-45

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(Received July 29, 1964)

This work adjoins that of Sobolev [1] in which was investigated the stability of fundamental (unperturbed) motion of a symmetric closed vessel with a single fixed point (symmetric top), which was filled with an ideal incompressible liquid. Here, the fundamental motion denotes a uniform free rotation in the gravitational field of the entire system (top + liquid) as a single rigid body about the top's axis of symmetry which does not alter its position in space, while the cause of perturbation is a small deviation of this axis from its initial position and the simultaneous "inclusion" of other external forces. The paper [1] investigates the stability of two degrees of freedom of the top characterized by the two-coordinate projection of the top's unit vector along its axis upon a plane perpendicular to the axis of rotation for unperturbed motion. Also, it is implicitly assumed that the third degree of freedom has no effect upon these two degrees of freedom, and that consequently, such a separate investigation of stability along these two coordinates is valid. Section 1 of the present paper justifies this assumption. The exposition of this section is parallel to the corresponding part in [1] with natural additions associated with the introduction of a new degree of freedom which is the angular velocity of the top's characteristic rotation. Section 2 investigates the stability of the third degree of freedom. It is shown that in difference with the first two coordinates, the stability of which depends on the form of the top's shell and the physical parameters of the problem, the angular velocity is always stable.

In conclusion, the author thanks S.L. Sobolev for his interest and valuable advice.

1. 1. Consideration is given to a heavy symmetric top fixed at the foot and completely filled with an ideal, incompressible fluid of density ρ. The top is rotating with a constant angular velocity ω about the axis of symmetry of order $k > 2$ (the order of symmetry is determined as the lowest natural number n such that the top coincides with itself for a rotation angle of $2\pi/k$ about its axis). Let S be the surface of the top's cavity filled with a liquid; M_1 and M_2 are the masses of the top and the liquid, respectively; C_1 and C_2 are the moments of inertia of the top and the liquid about the axis of symmetry; A_1 and A_2 are the moments of inertia of the top and the liquid about axes perpendicular to the axis of symmetry; l_1 and l_2

are the distances from the fixed foot of the top to the centers of gravity of the top and that of the liquid, and g is the gravitational açceleration.

The origin of the fixed Cartesian coordinate system $Ox^*y^*z^*$ is attached to the fixed point of the top, the z^*-axis being directed vertically upward. The axis of the top in unperturbed motion will also be assumed directed upward along the vertical. The spatial orientation of the top is usually given by three Euler angles. However, for the stability investigation of small deviations of the top's axis, the position of the top is more conveniently determined by the parameters X^*, Y^*, ϑ, in which X^* and Y^* denote the coordinates of the projection on a plane x^*y^* of a unit vector directed along the top's axis from the foot to the center of gravity. The parameter ϑ is related to the projection η of the top's angular velocity on its axis by the relationship $d\vartheta = \eta dt$.

The quantity $d\vartheta$ as is known, is not a total differential and therefore, η is not a Lagrangian generalized coordinate in the usual sense; it is a so-called quasi-coordinate. The parameter ϑ, has no definite meaning. A definite meaning have $d\vartheta$, or the projection η which is connected by known relationships with the Euler angles and which characterizes the so-called characteristic rotation of the top (rotation about the top's characteristic axis).

2. The equations of motion for the top in terms of the parameters X^*, Y^* and η under the assumption of small X^*, Y^* and $\eta - \omega$ are of the form

$$\begin{gathered} A_1X^{*\cdot\cdot} + C_1\omega Y^{*\cdot} - gl_1M_1X^* - M_{y^*}(p^*) - M_{y^*}{}^\circ = 0 \\ A_1Y^{*\cdot\cdot} - C_1\omega X^{*\cdot} - gl_1M_1Y^* + M_{x^*}(p^*) + M_{x^*}{}^\circ = 0 \\ C_1\eta^{\cdot} - M_{z^*}(p^*) - M_{z^*}{}^\circ = 0 \end{gathered} \qquad (1.1)$$

Here $M_{x^*}(p^*)$, $M_{y^*}(p^*)$ and $M_{z^*}(p^*)$ are the moment projections of fluid pressure forces acting on the shell of the top, $M_{x^*}{}^\circ$, $M_{y^*}{}^\circ$, $M_{z^*}{}^\circ$ are the moment projections of the external nongravitational forces. The system of equations (1.1) is incomplete since it includes the moment of fluid pressure forces which should be determined by the use of hydrodynamic equations

$$\frac{d\mathbf{u}^*}{dt} + \frac{1}{\rho}\operatorname{grad} p^* = \mathbf{F} - g\mathbf{k}, \qquad \operatorname{div} \mathbf{u}^* = 0 \qquad (1.2)$$

where $\mathbf{F}$ is the vector of external mass forces and $\mathbf{k}$ is the unit vector along the z^*-axis. A natural boundary condition for Equations (1.2) is the impermeability of the top's shell to fluid particles

$$u_n^*|_S = w_n^*|_S \qquad (1.3)$$

Here w_n^* is the normal component of the transport velocity of the top's shell dependent on the parameters X^*, Y^*, η.

The boundery condition therefore represents a feedback, and along with Equations (1.1) and the partial differential equations (1.2) yields a complete totality of relationships which fully determine the motion of the system of the top plus the fluid for arbitrary initial conditions. It would be too optimistic to expect an explicit expression for this motion. Such an objective is not set in this case. The objective, as noted previously, is to investigate the stability conditions in one sense or another, i.e. in establishing the conditions for which the motion corresponding to a small deviation from a free, uniform rotation about the vertical (*) will remain always small or bounded, or will not increase beyond a given order of magnitude with time dependent on the growth character of the external forces.

3. In investigating the effect of perturbations, it is natural to consider not the complete solution of the problem of perturbed motion but only the difference of this solution and the solution for unperturbed motion (*). This, apparently, requires the use of the coordinate system $Oxyz$, which rotates about the z^*-axis with angular velocity ω and which coincides initially with the fixed system $Ox^*y^*z^*$. Furthermore, instead of the scalar function p^* and the scalar η it is necessary to introduce another scalar function p (expressing the excess of fluid pressure, within the accuracy of an unessential constant, and resulting from the effect of the perturbation) and another scalar θ by the relationship

$$p^* = -\rho g z^* + {}^1/_2 \rho\omega^2 (x^{*2} + y^{*2}) + p, \qquad \eta = \omega + \theta \tag{1.4}$$

The acceleration of a fluid particle in the fixed system consists of the relative, transport and Coriolis accelerations. Utilizing this fact and (1.4), the equations of motion for the fluid in the rotating system of coordinates are obtained from (1.2) after neglecting the small terms of second order

$$\begin{aligned} &\frac{\partial u_x}{\partial t} - 2\omega u_y + \frac{1}{\rho}\frac{\partial p}{\partial x} = F_x, \qquad \frac{\partial u_z}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} = F_z \\ &\frac{\partial u_y}{\partial t} + 2\omega u_x + \frac{1}{\rho}\frac{\partial p}{\partial y} = F_y, \qquad \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \end{aligned} \tag{1.5}$$

Utilizing the relationship (1.4) it can be shown [1] that

$$\begin{aligned} M_{x^*}(p^*) &= M_{x^*}(p) - (gl_2M_2 + \omega^2 A_2 - \omega^2 C_2)\, Y^* \\ M_{y^*}(p^*) &= M_{y^*}(p) + (gl_2M_2 + \omega^2 A_2 - \omega^2 C_2)\, X^* \\ M_{z^*}(p^*) &= M_{z^*}(p) \end{aligned}$$

Substituting these expressions into Equations (1.1) and introducing the complex parameter $Z = X + iY = e^{-i\omega t}(X^* + iY^*)$, we get the complex form of the top's equations of motion in the rotating system of coordinates

$$A_1 Z^{\cdot\cdot} - (C_1 - 2A_1)\, i\omega Z^{\cdot} + L\omega^2 Z + 2iN(p) + 2iN^\circ = 0$$

$$C_1\theta^{\cdot} - M_z(p) - M_z^\circ = 0 \tag{1.6}$$

Here

$$L = C_1 + C_2 - A_1 - A_2 - \frac{g(l_1M_1 + l_2M_2)}{\omega^2}$$

$$2N(p) = M_x(p) + iM_y(p), \qquad 2N^\circ = M_x^\circ + iM_y^\circ$$

Let us introduce

$$\mu = z(\cos nx + i\cos ny) - (x + iy)\cos nz, \quad \nu = x\cos ny - y\cos nx \tag{1.7}$$

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*) The free ($F = M_{x^*}^\circ = M_{y^*}^\circ = M_{z^*}^\circ = 0$) and uniform ($\eta = \omega = \text{const}$) rotation about the vertical (unperturbed motion) has, as can be easily shown, the simple complete solution

$$X^* = Y^* = 0, \ \eta = \omega, \ \mathbf{U}^* = \omega\,\mathbf{k}\times\mathbf{r}, \ p^* = {}^1/_2\,\rho\omega^2(x^{*2} + y^{*2}) - \rho g z^* + \text{const}.$$

Then we get

$$2N\ (p) = i \iint \mu p\, dS, \qquad M_z\ (p) = \iint \nu p\, dS \tag{1.8}$$

Now let us express conveniently the boundary condition (1.3). The top's shell participates in two rotational motions caused by the deviation of the axis and its characteristic rotation. It is not difficult to see that the vector of angular velocity of the first motion is equal to $(-Y^{\cdot},\ X^{\cdot},\ 0)$ in the rotating coordinate, while the second motion has the angular velocity $(0,\ 0,\ \theta)$ with the accuracy of up to the infinitely small terms of second order.

Consequently, the transport velocity of the top's shell is expressed by the vector $(X^{\cdot}z - \theta y,\ Y^{\cdot}z + \theta x,\ - X^{\cdot}x - Y^{\cdot}y)$, and condition (1.3) yields the relation for the points on the surface S

$$u_n = X^{\cdot}\ (z \cos nx - x \cos nz) + Y^{\cdot}\ (z \cos ny - y \cos nz) + \\ + \theta\ (x \cos ny - y \cos nx)$$

For convenience, let us express this condition by Z and θ. We have

$$u_n = {}^1/_2\,(Z^{\cdot}\,\bar{\mu} + \bar{Z}^{\cdot}\mu) + \theta\nu \tag{1.9}$$

where μ and ν are determined by the equalities (1.7).

We introduce the complex variable $\zeta = x + iy$ and the complex functions

$$u_\zeta = u_x + iu_y, \quad u_{\bar\zeta} = u_x - iu_y, \quad F_\zeta = F_x + iF_y, \quad F_{\bar\zeta} = F_x - iF_y \tag{1.10}$$

Defining further the formal differentiation with respect to ζ and $\bar\zeta$ by the equalities

$$\frac{\partial}{\partial\zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial\bar\zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$$

it is easy to express the system (1.5) in a complex form. Thus

$$\begin{aligned} &\frac{\partial u_\zeta}{\partial t} + 2i\omega u_\zeta + \frac{2}{\rho}\,\frac{\partial p}{\partial\bar\zeta} = F_\zeta, \qquad &&\frac{\partial u_z}{\partial t} + \frac{1}{\rho}\,\frac{\partial p}{\partial z} = F_z \\ &\frac{\partial u_{\bar\zeta}}{\partial t} - 2i\omega u_{\bar\zeta} + \frac{2}{\rho}\,\frac{\partial p}{\partial\zeta} = F_{\bar\zeta}, \qquad &&\frac{\partial u_\zeta}{\partial\zeta} + \frac{\partial u_{\bar\zeta}}{\partial\bar\zeta} + \frac{\partial u_z}{\partial z} = 0 \end{aligned} \tag{1.11}$$

The expediency of complex variables is manifested in the possibility of representing the complete solution of the combined motion of the top and the fluid as a sum of k particular solutions (recalling that k is the order of symmetry of the top). Also, only two of the k sloutions need to be investigated since in the remaining solutions the interaction of the top and the fluid is absent. Futhermore, it appears that these solutions, in turn, can be investigated separately since in one of them only the Z parameter of the top participates along with the fluid parameters (the θ parameter is absent), and in the second solution only the θ parameter is included (Z is absent).

Decomposition of the complete solution into k solutions occurs as follows (see [1]). Let $\varphi(x,\ y,\ z;\ t)$ be an arbitrary (complex valued) function defined for $t \geqslant 0$ in a region V bounded by the surface S. Let us regard z and t as parameters with φ a function of two real variables x and y. Each such function φ will be connected with a function of a pair of complex variables ζ and $\bar{\zeta}$, for simplicity denoted by the same symbol φ, and defined by the equality

$$\varphi(\zeta, \bar{\zeta}) = \varphi(1/_2(\zeta+\bar{\zeta}); -1/_2 i(\zeta-\bar{\zeta}); z, t).$$

The two-dimensional complex manifold on which the function $\varphi(\zeta, \bar{\zeta})$ is defined is given by the condition $(x, y, z) \in V$. In view of the assumed symmetry of the region V, it is apparent that for any pair $(\zeta, \bar{\zeta})$ in the region where $\varphi(\zeta, \bar{\zeta})$ is defined there is a pair $(\zeta \exp(2\pi il/k), \bar{\zeta}\exp(-2\pi il/k))$ for any integral l. Let us consider now the k new functions

$$\varphi_{(s)}(\zeta, \bar{\zeta}) = \frac{1}{k}\sum_{l=0}^{k-1} \exp\frac{2\pi ils}{k}\,\varphi\left(\zeta\exp\frac{2\pi il}{k},\ \bar{\zeta}\exp\frac{-2\pi il}{k}\right) \quad (s=0, 1, \ldots, k-1)$$

It is apparent that $\varphi_{(s)}$ can also be defined for all integral s by setting $\varphi_{(s_1)} = \varphi_{(s_2)}$ for $s_1 \equiv s_2 \pmod{k}$. Also, in particular for the real φ, we get $\varphi_{(-s)} = \varphi_{(s)}$.

The following expansion is easily shown to be valid

$$\varphi(\zeta, \bar{\zeta}) = \sum_{s=0}^{k-1}\varphi_{(s)}(\zeta, \bar{\zeta})$$

The uniqueness of such an expansion is obvious: if $\varphi \equiv 0$, then $\varphi_{(s)} \equiv 0$ for any s.

The functions $\varphi_{(s)}$ possess the distinctive periodicity

$$\varphi_{(s)}\left(\zeta\exp\frac{2\pi i}{k},\ \bar{\zeta}\exp\frac{-2\pi i}{k}\right) = \exp\frac{-2\pi is}{k}\varphi_{(s)}(\zeta, \bar{\zeta})$$

It is easy to prove by direct calculation the validity of the converse statement: if a certain function φ possesses periodicity in the stated sense with period s, then $\varphi_{s'} = 0$ for $s' \neq s$ and $\varphi_{(s)} = \varphi$.

5. Let us now apply the (s) operation to Equations (1.11). Taking into consideration the easily verifiable relationships

$$\frac{\partial\varphi_{(s)}}{\partial\zeta} = \left(\frac{\partial\varphi}{\partial\zeta}\right)_{(s+1)}, \qquad \frac{\partial\varphi_{(s)}}{\partial\bar{\zeta}} = \left(\frac{\partial\varphi}{\partial\bar{\zeta}}\right)_{(s-1)} \qquad (s=0,1,\ldots, k-1)$$

we get

$$\frac{\partial u_{\zeta,(s-1)}}{\partial t} + 2i\omega u_{\zeta,(s-1)} + \frac{2}{\rho}\frac{\partial p_{(s)}}{\partial\bar{\zeta}} = F_{\zeta,(s-1)}$$

$$\frac{\partial u_{\bar{\zeta},(s+1)}}{\partial t} - 2i\omega u_{\bar{\zeta},(s+1)} + \frac{2}{\rho}\frac{\partial p_{(s)}}{\partial\zeta} = F_{\bar{\zeta},(s+1)} \tag{1.12}$$

$$\frac{\partial u_{z,(s)}}{\partial t} + \frac{1}{\rho}\frac{\partial p_{(s)}}{\partial z} = F_{z,(s)} \qquad \frac{\partial u_{\zeta,(s-1)}}{\partial\zeta} + \frac{\partial u_{\bar{\zeta},(s+1)}}{\partial\bar{\zeta}} + \frac{\partial u_{z,(s)}}{\partial z} = 0$$

The system (1.11) is therefore divided. Instead of a single system, we get k new systems relating $u_{\zeta,(s-1)}$, $u_{\bar{\zeta},(s+1)}$, $u_{z,(s)}$, $p_{(s)}$.

Let us turn now to the boundary condition. Substituting the expressions for u_x and u_y from (1.10) into (1.9) we get the following relationship on S:

$$\begin{gathered}\bar{\lambda}u_\zeta + \lambda u_{\bar{\zeta}} + 2u_z \cos nz = Z^{\cdot}\bar{\mu} + \bar{Z}^{\cdot}\mu + 2\theta\nu \\ \lambda = \cos nx + i \cos ny\end{gathered} \tag{1.13}$$

Noting that

$$\begin{aligned}\lambda\left(\zeta \exp\frac{2\pi i}{k},\quad \bar{\zeta}\exp\frac{-2\pi i}{k}\right) &= \exp\frac{2\pi i}{k}\lambda(\zeta, \bar{\zeta}) \\ \bar{\lambda}\left(\zeta \exp\frac{2\pi i}{k},\ \bar{\zeta}\exp\frac{-2\pi i}{k}\right) &= \exp\frac{-2\pi i}{k}\bar{\lambda}(\zeta, \bar{\zeta}) \\ \mu\left(\zeta \exp\frac{2\pi i}{k},\ \bar{\zeta}\exp\frac{-2\pi i}{k}\right) &= \exp\frac{2\pi i}{k}\mu(\zeta, \bar{\zeta}) \\ \bar{\mu}\left(\zeta \exp\frac{2\pi i}{k},\ \bar{\zeta}\exp\frac{-2\pi i}{k}\right) &= \exp\frac{-2\pi i}{k}\bar{\mu}(\zeta, \bar{\zeta}) \\ \nu\left(\zeta \exp\frac{2\pi i}{k},\ \bar{\zeta}\exp\frac{-2\pi i}{k}\right) &= \nu(\zeta, \bar{\zeta})\end{aligned}$$

we obtain $\lambda = \lambda_{(-1)}$, $\bar{\lambda} = \bar{\lambda}_{(1)}$, $\mu = \mu_{(-1)}$, $\bar{\mu} = \bar{\mu}_{(1)}$ and $\nu = \nu_{(0)}$. Therefore, applying the (s) operation to both parts of the equality (1.13) we get the following condition on S :

$$\bar{\lambda}u_{\zeta,\,(s-1)} + \lambda u_{\bar{\zeta},\,(s+1)} + 2u_{z(s)}\cos nz =$$

$$= Z^{\cdot}\bar{\mu}_{(s)} + \bar{Z}^{\cdot}\mu_{(s)} + 2\theta\nu_{(s)} = \begin{cases} Z^{\cdot}\bar{\mu}, & s = 1 \\ \bar{Z}^{\cdot}\mu & s = -1 \equiv k-1 \pmod{k} \\ 2\theta\nu & s = 0 \\ 0 & \text{for other } s\end{cases} \tag{1.14}$$

Substituting into $M_z(p)$ the expression p , we get, in accordance with (1.8),

$$M_z(p) = \sum_{s=0}^{k-1}\iint p_{(s)}\,\nu ds \tag{1.15}$$

We make a change of variables in each of the integrals in (1.15) and rotate the coordinate axes in the xy plane through an angle $2\pi/k$. Also in view of the symmetry of the top, the region of integration will not change. In accordance with the above indicated property of the φ function periodicity, the $p_{(s)}$ function will transpose into $\exp(-2s\pi i/k)\,p_{(s)}$, while the ν function will not change since $\nu = \nu_{(0)}$, as was shown previously. The Jacobian of this transformation is apparently equal to unity. Therefore,

$$\iint p_{(s)}\,\nu dS = \exp\frac{-2s\pi i}{k}\iint p_{(s)}\,\nu dS$$

and consequently, all integrals in (1.15) vanish except the integral in which $s = 0$. It is shown analogously that the integral on S from $p_{(s)}\mu$ is equal to zero if $s \neq 1$. Thus, we get

$$M_z(p) = M_z(p_{(0)}), \qquad N(p) = N(p_{(1)}) \tag{1.16}$$

6. Now it remains only to draw the conclusions. The system of equations (1.6) which describes the motion of the top contains the p function for the determination of which were invoked the relationships (1.12) and (1.14). However, according to (1.6) and (1.16) the motion of the top is affected only by the components $p_{(0)}$ and $p_{(1)}$, which can be determined from (1.12) and

(1.14) for $s = 0$ and $s = 1$. Conversely, the top affects only those components of fluid motion which are related by the relationships (1.12) and (1.14) for the same values of s since for other values of s the boundary condition (1.14) is homogeneous and does not contain the parameters of the top. Furthermore, the indicated relationships show that the parameter Z is affected only by the component $p_{(1)}$, while the system of relationships determining $p_{(1)}$, does not contain the θ parameter. Conversely, the θ parameter is affected only by the component $p_{(0)}$, which is determined by considering the relations (1.12) and (1.14) for $s = 0$ in which Z does not appear. Thus, only the cases (*) $s = 0$ and $s = 1$ need to be investigated These cases are not connected to each other and can be investigated separately. The case of $s = 1$ has been studied in [1]. The stability for $s = 0$, however, (the stability of the θ parameter) is proved in the following.

2. 1. In order to transfer the top effect from the boundary conditions into the equations, let us introduce the following new functions

$$v_x = u_{\bar{\zeta},(1)} + u_{\zeta,(-1)} - 2\theta \frac{\partial\psi}{\partial x}, \quad v_y = i\,(u_{\bar{\zeta},(1)} - u_{\zeta,(-1)}) - 2\theta \frac{\partial\psi}{\partial y}$$

$$v_z = 2u_{z,(0)} - 2\theta \frac{\partial\psi}{\partial z}$$

where the ψ function is defined by the conditions

$$\Delta\psi = 0, \qquad \frac{\partial\psi}{\partial n}\Big|_S = \nu \tag{2.1}$$

In accordance with (1.14) (for $s = 0$), the $\mathbf{v}$ vector satisfies the homogeneous boundary condition $v_n = 0$ on S. And the equation for $\mathbf{v}$ is obtained from Equations (1.12) for $s = 0$ by addition and subtraction. Thus, we have

$$\begin{gathered}\frac{\partial v_x}{\partial t} - 2\omega v_y + \frac{2}{\rho}\frac{\partial p_{(0)}}{\partial x} + 2\theta^{\cdot}\frac{\partial\psi}{\partial x} - 4\omega\theta\frac{\partial\psi}{\partial y} = G_x\\ \frac{\partial v_y}{\partial t} + 2\omega v_x + \frac{2}{\rho}\frac{\partial p_{(0)}}{\partial y} + 2\theta^{\cdot}\frac{\partial\psi}{\partial y} + 4\omega\theta\frac{\partial\psi}{\partial x} = G_y\\ \frac{\partial v_z}{\partial t} + \frac{2}{\rho}\frac{\partial p_{(0)}}{\partial z} + 2\theta^{\cdot}\frac{\partial\psi}{\partial z} = G_z, \quad \operatorname{div}\mathbf{v} = 0, \quad v_n|_S = 0\end{gathered} \tag{2.2}$$

where

$$G_x = F_{\bar{\zeta},(1)} + F_{\zeta,(-1)}, \qquad G_y = i\,(F_{\bar{\zeta},(1)} - F_{\zeta,(-1)}), \qquad G_z = 2F_{z,(0)}$$

The system of equations (2.2) along with the second equation of system (1.6), in which according to (1.16) the $M_z(p)$ should be replaced by $M_z(p_{(0)})$, yields a closed system of relationships which will be termed the D_θ system. Before investigating the D_θ system, we will make the following remark.

If it is regarded that the vector $\mathbf{G}$ is continuous in the region V up to the boundary and that it has continuous derivatives of first order inside V, then without loss of generality it can be stated that it is of a certain special form, namely

$$\mathbf{G} = \varphi(t)\operatorname{grad}\psi + \Psi(x, y, z; t)$$

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*) It can be easily seen that the solution of the problem for $s = k-1$ will be simply a complex-conjugate to the solution for $s = 1$.

where φ is a certain (known) function only of t, ψ is the solution of the boundary value problem (2.1), and the vector Ψ satisfies the conditions

$$\operatorname{div} \Psi = 0, \quad \Psi_n = 0 \quad \text{on } S \tag{2.3}$$

In order to prove this statement, we will denote by $\Phi(x, y, z; t)$ the solution of the Neumann problem with the boundary condition $\partial\Phi/\partial n = G_n$ on S for the Poisson's equation (in space variables) with the right-hand side equal to div $\mathbf{G}$.. It is obvious, here, that $\Psi = G - \operatorname{grad} \Phi$ satisfies both conditions (2.3). Consequently, it is only necessary to show that the Φ function can be considered equal to $\varphi(t)\psi(x,y,z)$. Denoting,

$$\varphi(t) = \frac{1}{\alpha^2}\iint \Phi \nu dS, \qquad \alpha^2 = \iiint\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] dV \tag{2.4}$$

we get the expansion $\Phi = f(x, y, z; t) + \varphi(t)\psi(x, y, z)$, where the f function satisfies the condition

$$\iint f\nu \, dS = 0$$

It remains to be noted that without loss of generality one may regard $f \equiv 0$, since the particular solution of the considered (linear) system D_θ, corresponding to the case of $f \not\equiv 0$, $\varphi \equiv 0$, $\Psi \equiv 0$, $M_z^0 \equiv 0$, is found by elementary means and is of the simple form $\theta \equiv 0$, $\mathbf{v} \equiv 0$, $p_{(0)} = {}^1/_2\,\rho f$. Thus the given statement is fully proved.

Let us introduce now instead of the function $p_{(0)}$ the function

$$q = 2p_{(0)} + 2\rho\theta^{\cdot}\psi - \rho\varphi\psi \tag{2.5}$$

Then the equations of system (2.2) will become

$$\begin{gathered}
\frac{\partial v_x}{\partial t} - 2\omega v_y + \frac{1}{\rho}\frac{\partial q}{\partial x} - 4\omega\theta\frac{\partial\psi}{\partial y} = \Psi_x \\
\frac{\partial v_y}{\partial t} + 2\omega v_x + \frac{1}{\rho}\frac{\partial q}{\partial y} + 4\omega\theta\frac{\partial\psi}{\partial x} = \Psi_y \\
\frac{\partial v_z}{\partial t} + \frac{1}{\rho}\frac{\partial q}{\partial z} = \Psi_z, \qquad \operatorname{div}\mathbf{v} = 0, \quad v_n|_S = 0
\end{gathered} \tag{2.6}$$

Multiplying the first three equations of system (2.6) by $\partial\psi/\partial x$, $\partial\psi/\partial y$ and $\partial\psi/\partial z$ respectively, adding and intergrating with respect to V , we obtain on the strength of (2.1), (2.3) and (2.6)

$$2\omega\iiint\left(v_x\frac{\partial\psi}{\partial y} - v_y\frac{\partial\psi}{\partial x}\right) dV + \frac{1}{\rho}\iint q\nu dS = 0$$

The obtained equation in conjuction with the second equation of system (1.6) and the equalities (1.16) and (2.5) gives

$$(C_1 + \rho\alpha^2)\,\theta^{\cdot} + \rho\omega\iiint\left(v_x\frac{\partial\psi}{\partial y} - v_y\frac{\partial\psi}{\partial x}\right) dV = M_z{}^\circ + \frac{\varphi\rho\alpha^2}{2} \tag{2.7}$$

Here α is determined by (2.4). The system of equations (2.6) and (2.7) is in convenient form of the system D_θ for its further investigation. It is easy to see that all quantities in the system D_θ are real.

Taking the div operator from the first three equations in (2.6) we get after addition

$$\Delta q = 2\omega\rho\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \tag{2.8}$$

The equations of the system (2.6) also yield the boundary condition on S

$$\frac{\partial q}{\partial n} = 2\rho\omega\,(v_y \cos nx - v_x \cos ny) + 4\omega\theta\rho\left(\frac{\partial\psi}{\partial y}\cos nx - \frac{\partial\psi}{\partial x}\cos ny\right) \tag{2.9}$$

It follows that the function q is determined (with accuracy of up to the unessential constant term) by specifying $\mathbf{v}$ and θ. Consequently, knowing $\mathbf{v}$ and θ (and also the external behavior, the functions Ψ, φ and M_z°) at any time, one can compute $\mathbf{v}_1 = d\mathbf{v}/dt$ and $\theta_1 = d\theta/dt$ in accordance with Equations (2.6) and (2.7) which thus specify a certain operator transforming the pair $(\mathbf{v}, \theta)$ into the pair $(\mathbf{v}_1, \theta_1)$. The exact determination of this operator is given in the following.

2. Let H denote a Hilbert space of real vector functions $\mathbf{v}$ defined in the region V and satisfying the conditions

$$(a) \qquad \iiint (v_x^2 + v_y^2 + v_z^2)\, dV = \|\mathbf{v}\|^2 < +\infty$$

(b) For any function $\varphi(x, y, z)$ which has inside the region V continuous first order derivatives the squares of which are integrable in V, the following identity is valid

$$\iiint \left(\frac{\partial\varphi}{\partial x} v_x + \frac{\partial\varphi}{\partial y} v_y + \frac{\partial\varphi}{\partial z} v_z\right) dV = 0$$

N o t e . For smooth $\mathbf{v}$ functions the condition (b) is equivalent to the fulfillment of the last two equalities in system (2.6).

Lemma (*). The linear manifold of $\mathbf{v}$ functions having derivatives of any order continuous up to the boundary of the region V is everywhere dense in H.

Let, furthermore, $\{R\}$ denote the real vector space of the elements $R = (\mathbf{v}, \theta)$ $(\mathbf{v} \in H$, θ is a real number) with a natural determination of linear operations and a norm given by the equality $\|R\| = \max\{\|\mathbf{v}\|, |\theta|\}$.

3. The equations of the system D_θ (Equations (2.6) and (2.7)) can be expressed in the form

$$R^{\cdot} = TR + R_0, \qquad R^{\cdot} = (d\mathbf{v}/dt, d\theta/dt) \tag{2.10}$$

Here dv/dt is understood in the sense of a strong convergence in H of the corresonding difference relationship, R_0 is the element of space $\{R\}$ having the components

$$\left(\Psi, \frac{2M_z^\circ + \rho\varphi\alpha^2}{2(C_1 + \rho\alpha^2)}\right) \tag{2.11}$$

and T is a linear operator presently defined only for those elements of space $\{R\}$ in which $\mathbf{v}$ has continuous derivatives. The set of such elements, in view of the above formulated Lemma, is dense everywhere in $\{R\}$. Therefore, if the operator T becomes bounded in this set it can be extended uniquely over the entire space $\{R\}$. Before showing the boundedness of T, we will write out the formulas defining this operator. If one lets $TR = R_1 = (\mathbf{v}_1, \theta_1)$ then, according to (2.6) and (2.7), we get

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*) The proof of this Lemma is contained in [2].

$$v_{1x} = 2\omega v_y - \frac{1}{\rho}\frac{\partial q}{\partial x} + 4\omega\theta\frac{\partial\psi}{\partial y}, \qquad v_{1y} = -2\omega v_x - \frac{1}{\rho}\frac{\partial q}{\partial y} - 4\omega\theta\frac{\partial\psi}{\partial x}$$

$$v_{1z} = -\frac{1}{\rho}\frac{\partial q}{\partial z}, \qquad \theta_1 = \frac{\rho\omega}{C_1+\rho\alpha^2}\iiint\left(v_y\frac{\partial\psi}{\partial x} - v_x\frac{\partial\psi}{\partial y}\right)dV \qquad (2.12)$$

These formulas should also be augmented by the relationships (2.8) and (2.9) which determine the function q in terms of the element R. From the the previous Note, it follows easily that $TR \in \{R\}$ for $R \in \{R\}$.

The boundedness of the operator T can be proved.

Let $\mathbf{v}$ be a sufficiently smooth vector. The first equation (2.12) yields the equality

$$v_{1x} + \frac{1}{\rho^2}\left(\frac{\partial q}{\partial x}\right)^2 + \frac{2}{\rho}v_{1x}\frac{\partial q}{\partial x} = 4\omega^2 v_y^2 + 16\omega^2\theta^2\left(\frac{\partial\psi}{\partial y}\right)^2 + 16\omega^2\theta v_y\frac{\partial\psi}{\partial y}$$

The second and third equations of the system (2.12) yield the similar equalities. Summing all three equalities and integrating over the region V, we get

$$\|\mathbf{v}_1\|^2 + \frac{1}{\rho^2}\|\operatorname{grad} q\|^2 + \frac{2}{\rho}\iiint\left(v_{1x}\frac{\partial q}{\partial x} + \right.$$

$$\left. + v_{1y}\frac{\partial q}{\partial y} + v_{1z}\frac{\partial q}{\partial z}\right)dV = 4\omega^2\iiint(v_x^2 + v_y^2)\,dV + 16\omega^2\theta^2\iiint\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \right.$$

$$\left. + \left(\frac{\partial\psi}{\partial y}\right)^2\right]dV + 16\omega^2\theta\iiint\left(v_x\frac{\partial\psi}{\partial x} + v_y\frac{\partial\psi}{\partial y}\right)dV$$

Let us eliminate the second term in the left-hand side of this equality. The next term is equal to zero in view of the property (b) in the definition of the space H. On the strength of the same property the last integral on the right-hand side can be replaced by an integral over V from $-v_x\,(\partial\psi/\partial z)$. Taking the above into consideration, we obtain the following bounds:

$$\|\mathbf{v}_1\| \leqslant 2\omega\,(\|\mathbf{v}\| + 2\,|\,\theta\,|\,\alpha) \leqslant 2\omega\,(1 + 2\alpha)\,\|R\| \qquad (2.13)$$

The last equality of system (2.12) yields the bounds

$$|\,\theta_1\,| \leqslant \frac{2\,\rho\omega\alpha}{C_1+\rho\alpha^2}\|\mathbf{v}\| \leqslant \frac{2\,\rho\omega\alpha}{C_1+\rho\alpha^2}\|R\| \qquad (2.14)$$

From (2.13) and (2.14), one gets, apparently, the following bounds for the norm of the operator

$$\|T\| \leqslant \max\left\{2\omega\,(1+2\alpha),\ \frac{2\rho\omega\alpha}{C_1+\rho\alpha^2}\right\}$$

Thus, the operator T is bounded and it can be extended over the entire space $\{R\}$. It is also obvious that the smooth solutions of Equation (2.10) will be the solutions of the D_θ system, and conversely, the solutions of the D_θ system will be smooth solutions of Equation (2.10). The solutions of Equation (2.10) however, which have no classical derivatives can be regarded naturally as the generalized solutions of the D_θ system.

4. We continue the investigation of Equation (2.10). Let us first consider the corresponding homogeneous equation

$$R^{\cdot} = TR \qquad (2.15)$$

It can be easily shown by direct verification that the solution of Equation (2.15 is of the form

$$R\,(t) = e^{Tt}\,R^{(0)}, \qquad R^{(0)} = (\mathbf{v}\,|_{t=0},\, \theta\,|_{t=0}) \tag{2.16}$$

Here the operator e^{Tt} is understood in the sense of a strong convergence of the corresponding power series guaranteed by the boundedness of the operator T in view of the obvious inequality $\| T^m \| \leqslant \| T \|^m$ $(m = 1, 2, \ldots)$.

It can be shown that there exists in the entire space $\{R\}$ a given positive-definite bilinear form $Q\,(R^{(1)},\ R^{(2)})$, with respect to which the operator T is antisymmetric in the sense that

$$Q\,(TR^{(1)}, R^{(2)}) = -Q\,(R^{(1)}, TR^{(2)}) \tag{2.17}$$

In order to prove this, we will seek the form Q in the form

$$Q\,(R^{(1)},\ R^{(2)}) = Q_1 \iiint (v_x^{(1)} \quad v_x^{(2)} + v_y^{(1)}\, v_y^{(2)} + v_z^{(1)}\, v_z^{(2)})\, dV + Q_2 \theta^{(1)}\, \theta^{(2)}$$

where Q_1 and Q_2 are positive constants subject to determination.

Then, utilizing (2.12), we get

$$Q\,(TR^{(1)}, R^{(2)}) = 2\omega Q_1 \iiint (v_y^{(1)}\, v_x^{(2)} - v_x^{(1)}\, v_y^{(2)})\, dV + 4\omega Q_1 \theta^{(1)} \iiint \left(\frac{\partial \psi}{\partial y}\, v_x^{(2)} - \right.$$
$$\left. - \frac{\partial \psi}{\partial x}\, v_y^{(2)} \right) dV + \frac{\rho\omega\theta^{(2)}\, Q_2}{C_1 + \rho\alpha^2} \iiint \left(\frac{\partial \psi}{\partial x}\, v_y^{(1)} - \frac{\partial \psi}{\partial y}\, v_x^{(1)} \right) dV$$

$$Q\,(R^{(1)}, TR^{(2)}) = 2\omega Q_1 \iiint (v_x^{(1)}\, v_y^{(2)} - v_y^{(1)} v_x^{(2)})\ \ dV + 4\omega Q_1 \theta^{(2)} \iiint \left(\frac{\partial \psi}{\partial y}\, v_x^{(1)} - \right.$$
$$\left. - \frac{\partial \psi}{\partial x}\, v_y^{(1)} \right) dV + \frac{\rho\omega\theta^{(1)}\, Q_2}{C_1 + \rho\alpha^2} \iiint \left(\frac{\partial \psi}{\partial x}\, v_y^{(2)} - \frac{\partial \psi}{\partial y}\, v_x^{(2)} \right) dV$$

It is now obvious that the inequality (2.17) will be fulfilled, and Q_1 and Q_2 satisfy the condition $\rho Q_2 = 4\,(C_1 + \rho\alpha^2)\, Q_1$. As should have been expected, this condition defines Q_1 and Q_2 within the accuracy of a common multiplier. Letting, for example, $Q_1 = \frac{1}{4}\rho$, $Q_2 = C_1 + \rho\alpha^2$ we obtain the bilinear form satisfying condition (2.17).

From (2.17) we have $Q\,(T^m R^{(1)}, R^{(2)}) = (-1)^m Q\,(R^{(1)},\ T^m R^{(2)})\ (m = 1, 2, \ldots)$. Therefore, on the strength of the linearity of Q, each argument separately easily yields $Q\,(e^{Tt} R^{(1)}, R^{(2)}) = Q\,(R^{(1)}, e^{-Tt} R^{(2)})$. Consequently, the following relationship is valid:

$$Q\,(e^{Tt} R,\ e^{Tt} R) = Q\,(R, R), \quad R \in \{R\} \tag{2.18}$$

The stability of the solution of the homogeneous equation (2.15) follows easily from the derived relationship. Indeed, let us introduce in the space $\{R\}$ along with the original norm also the equivalent norm generated by the Q form.

$$\| R \|_1 = \sqrt{Q\,(R, R)} \tag{2.19}$$

At the same time the equality (2.18) shows that whatever the initial value of $R^{(0)}$, the solution of the Cauchy problem for Equation (2.15) defined by Formula (2.16), remains for all time on the surface of a fixed sphere with center at the origin (zero) and the radius equal to $\sqrt{Q\,(R^{(0)},\ R^{(0)})}$, i.e. $\| R\,(t) \|_1 = \| R^{(0)} \|_1$ for any time t. Thus, in the case of smallness of the values of $\|\mathbf{v}\|$ and $|\theta|$ for $t = 0$, the quantities $\|\mathbf{v}(t)\|$ and $|\theta(t)|$ remain small for all time. This indicates the stability of the equations of system

D_θ and, in particular, the stability of the parameter θ in the case of zero external nongravitational forces.

The equality (2.18) can also be interpreted somewhat differently. It shows that the positive-definite quadratic form $Q(R, R)$ is the integral of motion of Equation (2.15) since along the trajectory of this equation $dQ/dt = 0$ in view of (2.16) and (2.18).

Let us turn now to the nonhomogeneous equation (2.10). The solution of this equation, as can be verified directly, is of the form

$$R(t) = e^{Tt} R^{(0)} + \int_0^t e^{T(t-\tau)} R_0(\tau)\, d\tau \tag{2.20}$$

where the integral is understood in the sense of a strong convergence of the corresponding integral sum. Utilizing the equalities (2.18) and (2.19) and taking an elementary estimate, we obtain from (2.20)

$$\| R(t) \|_1 \leqslant \| R^{(0)} \|_1 + J(t), \qquad J(t) = \int_0^t \| R_0(\tau) \|_1\, d\tau \tag{2.21}$$

Thus, for the parameter θ (in contrast to the parameters χ and Y) there are no resonance phenomena such as a fast increase with t for specific relationships among the problem constants and the geometric form of the top shell. If, for example, $\|R_0(t)\|_1$ remains bounded for the entire period of time, then, according to (2.21), $\|R(t)\|_1$ (in particular $\theta(t)$) increases no faster than the first order of t. Furthermore, for small initial values of $\| R^{(0)}\|_1$, as well as the "summed action" of the external forces (quantity $J(\infty)$), then $\|R(t)\|_1$ (in particular $\theta(t)$) also remains small for the entire period of time. In this sense, it can be stated that the stability exists also in the general (nonhomogeneous: $\mathbf{F} \neq 0,\ M_z^\circ \neq 0$) case.

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Translated by V.C.